Columbia University
Department of Economics
Economics – GR5410: Mathematical Methods in Economics
Homework 4

This problem set (PS) is due on Wednesday, December 9 at the beginning of class. The maximum score is 100 points. Be sure to review the syllabus for details about homework assignments and their grading! **Feel free to contact me or Cesar or Vinayak via e-mail if you have specific questions about the HW assignment.**

Note that some Exercises have several parts, and each part may conceal more than one task for you. Be sure to answer every question **thoroughly** for full credit!

Exercise 1 (16 points)
Suppose that you repeatedly toss a fair coin:

(a) What is the probability that you get TT (two heads in a row) before HT?

(b) What is the probability that you get TT before TH? What is the expected number of tosses until you get HH for the first time?

(c) What is the expected number of tosses until you get HT for the first time?

Exercise 2 (20 points) This question connects Riemann integration to Lebesgue integration. Throughout, we consider the integral of a function f over a compact interval $[a, b]$.

(a) Suppose f is unbounded (on this interval), show it is not Riemann integrable.

(b) Suppose f is discontinuous at a positive measure set A, show it is not Riemann integrable. (Hint: For each n, let A_n be the set of points x^0 such that $limsup_{x \to x^0} f(x) \geq liminf_{x \to x^0} f(x) + \frac{1}{n}$. Argue that the sets A_n are increasing in n, with $lim_n A_n = A$ Thus some A_n has positive measure. Note that in any partition of $[a, b]$, if a subinterval contains some point $x^0 \in A_n$ then the sup and inf of $f(x)$ differ by at least $1/n$ on this subinterval. Argue why this implies that the upper and lower Riemann sums cannot converge to the same number.

(c) Conversely suppose f is bounded and continuous almost everywhere, show it is Riemann integrable.(Hint: Without loss of generality assume $[a, b] = [0, 1]$ and $0 \leq f \leq 1$. Fix any $\epsilon > 0$, it suffices to demonstrate a partition such that the upper and lower Riemann sums differ by no more than ϵ. Consider the partition $\mathbb{P}_n$ that divides $[0, 1]$ evenly into 2^n subintervals. Let B_n be the union of those subintervals in $\mathbb{P}_n$ such that the sup and inf of $f(x)$ differ by at least $\epsilon/2$ on each of these subintervals. Argue that the sets B_n are decreasing in n and that each point in the limit set $lim_n B_n$ is a point where f is discontinuous. Thus $m(lim_n B_n) = 0$, so that $m(B_n) \leq \epsilon/2$ for some n. Argue why this implies that $\mathbb{P}_n$ is a desired partition.

(d) Show that when f is Riemann integrable, it is also Lebesgue integrable, and the two forms of integration lead to the same answer. (Hint: argue that the Lebesgue integral is larger than every lower Riemann sum, and smaller than every upper Riemann sum)

Exercise 3 (16 points) A gambler starts with 0 dollars. On each day, he either wins an additional dollar, or loses all his money, with equal probabilities.

(a) Write down the Markov chain (state space and transition matrix) that keeps track of the gambler's total earnings at the end of each day.

(b) Is this Markov chain irreducible? Is it aperiodic?

(c) What is the distribution of the random variable that represents the first time that the gambler has no money again?

(d) Is the Markov chain recurrent, and is it positive-recurrent?

(e) Find the stationary distribution for this Markov chain.

(f) Verify that the distribution of X_n converges to the stationary distribution as $n \to \infty$. (Hint: prove by induction that the distribution of X_n agrees with the stationary distribution on the states $\{0, 1, 2, ..., n-1\}$.

Exercise 4 (16 points)

A machine has two parts, A and B. Each part is checked at the beginning of each day. If at least one of the two parts is working, the machine is in working order and neither part is replaced. If neither part is working, the machine is in repair for the day, and both parts are replaced with working parts. If both A and B are working, each will fail independently with probability 0.1 the next day. If A is working but B is not, then A fails with probability 0.2 the next day (and B continues to fail). And if B is working but A is not, then B fails with probability 0.3 the next day.

(a) Determine the transition matrix for the day-to-day status of the machine, on states 1, 2, 3, 4 which code (in order) the parts that are working: A and B, A but not B, B but not A, neither A nor B.

(b) Is this Markov chain irreducible? Is it aperiodic?

(c) Does this Markov chain admit a stationary distribution? If so, find it.

(d) What is the long-run fraction of days on which part A is working?

Exercise 5 (16 points) Suppose that two (standard) fair dice are rolled and their scores recorded.

(a) Describe the sample space Ω for the experiment and define the probability space associated with Ω.

(b) Define a random variable X as the sum of the values on the two dice. Determine the new probability space generated by the random variable X and describe the probability measure induced by X.

(c) Define a random variable U as the minimum score of the two dice, and the random variable V as the maximum score of the two dice. Determine the probability space generated by each of the random variables U and V. Describe the probability measure induced by U and by V.

(d) Determine the $Pr\{U = u, V = v\}$ for each pair (u, v) in the Cartesian product of the probability spaces you found in the previous part.

Exercise 6 (16 points)

A jar has 3 coins, one of which is fair, another is double-headed, and the last one is double-tailed. You first pick a random coin and toss it once. Suppose H comes up:

(a) Given this, what is the probability that the chosen coin is fair/double- headed/double-tailed?

(b) If you then pick another coin from the remaining 2 coins in the jar, what is the probability that the new coin is fair/double-headed/double-tailed?

(c) Afterwards you toss the second coin twice in a row. What is the probability that you see TT (again, conditional on the initial observation of H)?